

# The Marginally Stable Circular Orbit of the Fluid Disk around a Black Hole

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(Dated: June 10, 2016)

The inner boundary of a black hole accretion disk is often set to the marginally stable circular orbit (or the innermost stable circular orbit, ISCO) around the black hole. It is important for the theories of black hole accretion disks and their applications to astrophysical black hole systems. Traditionally, the marginally stable circular orbit is obtained by considering the equatorial motion of a test particle around a black hole. However, in reality the accretion flow around black holes consists of fluid, in which the pressure often plays an important role. Here we consider the influence of fluid pressure on the location of marginally stable circular orbit around black holes. It is found that when the temperature of the fluid is so low that the thermal energy of a particle is much smaller than its rest energy, the location of marginally stable circular orbit is almost the same as that in the test particle case. However, we demonstrate that in some special cases the marginally stable circular orbit can be different when the fluid pressure is large and the thermal energy becomes non-negligible comparing with the rest energy. We present our results for both the cases of non-spinning and spinning black holes. The influences of our results on the black hole spin parameter measurement in X-ray binaries and the energy release efficiency of accretion flows around black holes are discussed.

PACS numbers: 04.70.-s, 97.60.Lf, 98.35.Mp

## I. INTRODUCTION

In the spacetime around a black hole, the circular motion of a test particle is not always possible according to general relativity. There is a smallest radius on which the circular motion of a test particle is marginally stable [1]. This radius is called marginally stable circular orbit. With a small perturbation, the circularly moving particle at this radius will then plunge into the black hole freely. The properties of the transition from inspiral to plunge depend on the mass ratio  $\eta$  of the particle and the black hole [2]. The transition would be less gradual (or more "abrupt") with smaller  $\eta$  (See also [3, 4]). This property is more relevant in the context of the inspiral in a binary black hole system, when the two black holes are just about to merge. When the self-force of the inspiralling particle is considered, the location of the marginally stable orbit is also modified [5].

The marginally stable circular orbit is also important in the estimate of energy release of the accretion to a black hole. By calculating the binding energy of the circular motion on this radius [6], one can estimate how much of the total energy can be released during the accretion process. For the accretion flow around a black hole, the energy release efficiency ranges from 5.6% (for non-spinning or Schwarzschild black holes) to 42% (for extreme Kerr black holes). The location of the marginally stable circular orbit and the energy release efficiency are closely related, both depending on the spin parameter of the black hole [7].

In some applications of accretion disk model in astrophysical observations, the location of marginally stable circular orbit is also crucial. Since the marginally stable circular orbit depends on the spin of the central black hole, it is possible to measure the spin of a black hole

if the corresponding marginally stable orbit can be measured. By fitting the observed soft state spectra of black hole X-ray binaries, which are assumed to come from a thin disk with its inner boundary at the marginally stable circular orbit, one can derive the location of this orbit and estimate the spin parameters of the black holes in X-ray binaries [8].

In the theory of accretion flow, the marginally stable circular orbit is also important. Due to its transitional nature from the inspiralling region to the plunging region, it is often believed that an accretion disk is torque free at this radius. Although still being debated when considering magnetic fields [9, 10], the torque free condition on the marginally stable circular orbit serves as an important inner boundary condition in many accretion disk models.

The self-gravity of the particle is not relevant in the context of accretion disks, but the material in a real accretion disk is fluid rather than test particles. When considering fluid, the particles within it are interacting with each other, and the pressure plays an important role in the dynamics. In the study of accretion disk theory, the marginally stable circular orbit for a test particle is used as the inner boundary of an accretion disk in many models. More precisely, it has been shown by analytical theory of fluid tori around black holes that the inner boundary of an accretion disk lies between the marginally stable circular orbit and the marginally bound orbit, while both expressions are for test particles [11–13]. In this work, however, we focus on a different problem, that is, we try to investigate the marginally stable circular orbit itself, with the influence from the pressure in the fluid. In section 2, we briefly mention the marginally stable circular orbit in the case of a test particle. Then we consider a thin disk consisting of perfect fluid around

a black hole and discuss the marginally stable circular orbit in this case in section 3. In Section 4 we present a brief discussion on our results. In this paper, we set  $c = G = M = 1$ , with  $c$ ,  $G$ , and  $M$  the speed of light, gravitational constant, and the mass of the black hole, respectively. We use  $(+ - - -)$  signature all through.

## II. MARGINALLY STABLE CIRCULAR ORBIT OF A TEST PARTICLE

The metric of the spacetime outside a Kerr (spinning, non-charged) black hole is [1]

$$ds^2 = \frac{\rho^2 \Delta}{A} dt^2 - \frac{A \sin^2 \theta}{\rho^2} (d\phi - \omega dt)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2, \quad (1)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ A &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \\ \Delta &= r^2 - 2r + a^2, \quad \omega = \frac{2ar}{A}, \end{aligned} \quad (2)$$

and  $a$  is the spin parameter of the black hole. The covariant form of the metric can be written as

$$\begin{aligned} g^{tt} &= -\frac{1}{\rho^2} \left[ a^2 \sin^2 \theta - \frac{(r^2 + a^2)^2}{\Delta} \right], \quad g^{rr} = -\frac{\Delta}{\rho^2}, \\ g^{\theta\theta} &= -\frac{1}{\rho^2}, \quad g^{\phi\phi} = -\frac{1}{\rho^2} \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right), \\ g^{t\phi} &= -\frac{a}{\rho^2} \left( 1 - \frac{r^2 + a^2}{\Delta} \right). \end{aligned} \quad (3)$$

For a steady and axis-symmetric spacetime (e.g. Schwarzschild, Kerr), there are two apparent constants of motion of a free particle with unit mass

$$u_t = E, \quad u_\phi = -L \quad (4)$$

Since the metric is block diagonal, the normalization condition of the 4-velocity can be written as

$$g^{tt} u_t u_t + 2g^{t\phi} u_t u_\phi + g^{\phi\phi} u_\phi u_\phi + g_{rr} u^r u^r + g^{\theta\theta} u^\theta u^\theta = 1 \quad (5)$$

Consider the circular orbital motion on the equatorial plane ( $\theta = \pi/2$ ,  $u^\theta \equiv d\theta/d\tau = 0$ ), the equation of motion can be written as

$$r^4 \left( \frac{dr}{d\tau} \right)^2 = U_r, \quad (6)$$

where

$$\begin{aligned} U_r &= (r^4 + a^2 r^2 + 2a^2 r) E^2 + 4a r E L - (r^2 - 2r) L^2 \\ &\quad - (r^4 - 2r^3 + a^2 r^2). \end{aligned}$$

(7)

In order to maintain a circular orbit,  $dr/d\tau$  and  $d^2 r/d\tau^2$  should both vanish, which imply

$$U_r = 0, \quad U'_r = 0, \quad (8)$$

where the prime denotes the derivative to  $r$ . The two constants of motion can be derived from the above two equations as

$$E = \frac{r^{3/2} - 2r^{1/2} + a}{r^{3/4}(r^{3/2} - 3r^{1/2} + 2a)^{1/2}}, \quad (9)$$

$$L = -\frac{r^2 - 2ar^{1/2} + a^2}{r^{3/4}(r^{3/2} - 3r^{1/2} + 2a)^{1/2}}. \quad (10)$$

For prograde motion,  $a > 0$ ; for retrograde motion,  $a < 0$ . Not all the circular orbits are stable. The stable ones should also fulfill  $U''_r \leq 0$ , and the marginal stable circular orbit corresponds to  $U''_r = 0$ . Using the expression for  $E$  and  $L$ , we can derive the marginal stable orbit

$$r_{ms} = 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}, \quad (11)$$

where

$$Z_1 \equiv 1 + (1 - a^2)^{1/3} [(1 + a)^{1/3} + (1 - a)^{1/3}],$$

$$Z_2 \equiv (3a^2 + Z_1^2)^{1/2}.$$

The upper sign and the lower sign are for prograde and retrograde motion of the particle, respectively. Note that for a Schwarzschild black hole,  $a = 0$ ,  $r_{ms} = 6$ . The expression above has been widely adopted in various astrophysical studies of black hole systems.

## III. MARGINALLY STABLE CIRCULAR ORBIT OF A PERFECT FLUID DISK

For perfect fluid, the energy-momentum tensor can be written as

$$T_\nu^\mu = (p + \varepsilon) u^\mu u_\nu + p \delta_\nu^\mu, \quad (12)$$

where  $u^\mu$ ,  $p$ , and  $\varepsilon$  are the 4-velocity, pressure, and the energy density respectively. It can be proved that there are also two constants of motion [13]:

$$\frac{p + \varepsilon}{n} u_t = \mathcal{E}, \quad (13)$$

and

$$\frac{p + \varepsilon}{n} u_\phi = -\mathcal{L}, \quad (14)$$

where  $n$  is the number density. Note that these two constants of motion represent no longer the energy and angular momentum. Use the normalization of the 4-velocity, it is easy to show that

$$r^4 \left( \frac{dr}{d\tau} \right)^2 = U_r, \quad (15)$$

where the effective potential

$$\begin{aligned} U_r = & \left( \frac{n}{p + \varepsilon} \right)^2 [(r^4 + a^2 r^2 + 2a^2 r) \mathcal{E}^2 + 4ar\mathcal{E}\mathcal{L} \\ & - (r^2 - 2r)\mathcal{L}^2] - (r^4 - 2r^3 + a^2 r^2). \end{aligned} \quad (16)$$

### A. The Schwarzschild Black Hole Case

We first consider the case of a thin fluid disk around a non-rotating black hole (with spin parameter  $a = 0$ ). The effective potential becomes

$$U_r = \left( \frac{n}{p + \varepsilon} \right)^2 [r^4 \mathcal{E}^2 - (r^2 - 2r)\mathcal{L}^2] - (r^4 - 2r^3). \quad (17)$$

In this case, the energy density  $\varepsilon$  consists of two parts: One is the rest energy (proportional to the number density  $n$ ) and the other one is the thermal energy (assumed to be proportional to the pressure  $p$ ). Therefore we have

$$\varepsilon = m_0 n + \frac{1}{\gamma - 1} p, \quad (18)$$

where  $m_0$  and  $\gamma$  are the rest mass of the particle and the ratio of specific heat, both of which are constants. If the dependence of both number density and pressure on the radius are of power law forms, we can parameterize the function  $n/(p + \varepsilon)$  as

$$\frac{n}{p + \varepsilon} = \frac{B}{1 + Cr^{-b}}, \quad (19)$$

where we have assumed  $p/\rho \propto T \propto r^{-b}$ , where  $\rho \equiv m_0 n$  is the density of the fluid. For the standard thin disk model [7, 14],  $b$  is  $3/8, 9/10, 3/4$ , in the inner, middle and outer regions, respectively. For the self-similar solution of an advection dominated accretion flow (ADAF, [15, 16]),  $b$  equals to 1. However, one should note that these dependence on radius  $r$  is only for the region where  $r \gg r_{ms}$ . When  $r \sim r_{ms}$ ,  $b$  can be negative if the torque free condition at  $r \sim r_{ms}$  is applied, which may be more relevant in the case of real accretion flows around black holes. The constant  $C$  is always positive.

Since the constant  $B$  can be absorbed into  $\mathcal{E}$  and  $\mathcal{L}$ , the effective potential can be rewritten as

$$U_r = \left( \frac{1}{1 + Cr^{-b}} \right)^2 [r^4 \mathcal{E}^2 - (r^2 - 2r)\mathcal{L}^2] - (r^4 - 2r^3). \quad (20)$$

In a normal fluid, the temperature is usually low and the thermal energy is much smaller than the rest energy, that is

$$\frac{1}{\gamma - 1} p \ll m_0 n. \quad (21)$$

In another word, the constant  $C$  is very small ( $C \ll 1$ ) in this case. Then there is not much influence from the pressure. The effective potential is approximately the same as that in the test particle case, which means the location of the marginally stable circular orbit is almost the same even in the fluid case.

However, in some hot accretion flows around black holes, e.g. ADAF [15, 16], the thermal energy may be relevant. We can estimate the ratio of thermal energy to rest energy in this case as

$$\theta_i \equiv \frac{kT_i}{m_i c^2} \approx 0.1, \quad (22)$$

$$\theta_e \equiv \frac{kT_e}{m_e c^2} \approx 0.1, \quad (23)$$

where  $k$ ,  $T_i$ ,  $m_i$ ,  $T_e$ ,  $m_e$ ,  $c$  are the Boltzmann constant, ion temperature, ion mass, electron temperature, and speed of light, respectively. In this case,  $C$  is not very large ( $C < 1$ ) but also non-negligible. We can expand the marginal stable circular orbit as

$$r_{ms} = r_{ms,0} + C\Delta r_{ms}. \quad (24)$$

In order to calculate the correction term  $\Delta r_{ms}$ , we have to know the first and second derivatives of the effective potential, Eq. (20). They are

$$\begin{aligned} U'_r = & \frac{2bCr^{-(b+1)}}{(1 + Cr^{-b})^3} [r^4 \mathcal{E}^2 - (r^2 - 2r)\mathcal{L}^2] \\ & + \frac{1}{(1 + Cr^{-b})^2} [4r^3 \mathcal{E}^2 - (2r - 2)\mathcal{L}^2] - (4r^3 - 6r^2) \end{aligned} \quad (25)$$

and

$$\begin{aligned} U''_r = & \frac{2b(b+1)Cr^{-(b+2)}}{(1 + Cr^{-b})^3} [r^4 \mathcal{E}^2 - (r^2 - 2r)\mathcal{L}^2] \\ & + \frac{6b^2C^2r^{-2(b+1)}}{(1 + Cr^{-b})^4} [r^4 \mathcal{E}^2 - (r^2 - 2r)\mathcal{L}^2] \\ & + \frac{4bCr^{-(b+1)}}{(1 + Cr^{-b})^3} [4r^3 \mathcal{E}^2 - (2r - 2)\mathcal{L}^2] \\ & + \frac{1}{(1 + Cr^{-b})^2} [12r^2 \mathcal{E}^2 - 2\mathcal{L}^2] - (12r^2 - 12r). \end{aligned} \quad (26)$$

Setting Eq. (20) and above two equations to 0 ( $U_r = U'_r = U''_r = 0$ ), we get an equation after eliminating  $\mathcal{E}$  and  $\mathcal{L}$ ,

$$(r-3)\left[-\frac{2b(b+1)Cr^{-b}}{1+Cr^{-b}}(r-2)-\frac{2b^2C^2r^{-2b}}{(1+Cr^{-b})^2}(r-2)\right. \\ \left.+\frac{4bCr^{-b}}{1+Cr^{-b}}(4r-6)\right]+(10r-24)\left[1-\frac{bCr^{-b}}{1+Cr^{-b}}(r-2)\right] \\ -12(r-3)=0. \quad (27)$$

Keeping the terms of the order  $O(C)$  we can get

$$\Delta r_{ms} = br_{ms,0}^{-b}\{(r_{ms,0}-3)[(7-b)r_{ms,0}+(2b-10)] \\ -(r_{ms,0}-2)(5r_{ms,0}-12)\} \quad (28)$$

For a Schwarzschild spacetime,  $r_{ms,0} = 6$ , so we get

$$r_{ms} = r_{ms,0} + C\Delta r_{ms} = 6 + 6^{-b}Cb(24 - 12b). \quad (29)$$

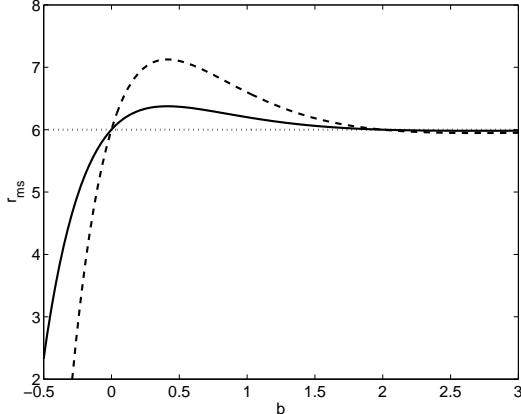


FIG. 1: The dependence of the marginally stable circular orbit  $r_{ms}$  in Schwarzschild spacetime on the power-law index  $b$  in the case of  $C < 1$ . The solid line and dashed line are for  $C = 0.1$  and  $C = 0.3$ , respectively. The dotted horizontal line represents  $r_{ms} = 6$  in the test particle case.

In the equation above,  $\Delta r_{ms}(b)$  has its maximum 3.75 at  $b = 0.413$  and local minimum  $-0.18$  at  $b = 2.7$ , respectively. So the maximum of the marginally stable circular orbit is

$$r_{ms,max} = 6 + 3.75C, \quad (30)$$

corresponding to  $b = 0.413$  and the local minimum is

$$r_{ms,min} = 6 - 0.18C, \quad (31)$$

corresponding to  $b = 2.7$ . When  $b \rightarrow \infty$ ,  $r_{ms} \rightarrow 6$ , and when  $b$  is negative,  $r_{ms} < 6$  and it decreases monotonically with the decreasing of  $b$  (see Fig. 1). Therefore, in this case when the thermal energy is smaller than the rest energy, the influence of the pressure on the marginally stable circular orbit is non-negligible, especially when  $b < 1.5$ .

## B. The Kerr Black Hole Case

In Kerr spacetime, the effective potential is expressed by Eq. (16). When  $C$  is small, we can do the same expansion as in Schwarzschild spacetime.

In Kerr spacetime, the effective potential, and its first and second order derivatives are

$$U_r = \left(\frac{1}{1+Cr^{-b}}\right)^2 [(r^4 + a^2r^2 + 2a^2r)\mathcal{E}^2 + 4ar\mathcal{E}\mathcal{L} \\ -(r^2 - 2r)\mathcal{L}^2] - (r^4 - 2r^3 + a^2r^2) \quad (32)$$

$$U'_r = \frac{2bCr^{-(b+1)}}{(1+Cr^{-b})^3}[(r^4 + a^2r^2 + 2a^2r)\mathcal{E}^2 + 4ar\mathcal{E}\mathcal{L} - (r^2 - 2r)\mathcal{L}^2] + \\ \frac{1}{(1+Cr^{-b})^2}[(4r^3 + 2a^2r + 2a^2)\mathcal{E}^2 + 4a\mathcal{E}\mathcal{L} - (2r - 2)\mathcal{L}^2] \\ - (4r^3 - 6r^2 + 2a^2r) \quad (33)$$

and

$$U''_r = -\frac{2b(b+1)Cr^{-(b+2)}}{(1+Cr^{-b})^3}[(r^4 + a^2r^2 + 2a^2r)\mathcal{E}^2 + 4ar\mathcal{E}\mathcal{L} \\ -(r^2 - 2r)\mathcal{L}^2] + \frac{6b^2C^2r^{-2(b+1)}}{(1+Cr^{-b})^4}[(r^4 + a^2r^2 + 2a^2r)\mathcal{E}^2 + 4ar\mathcal{E}\mathcal{L} \\ -(r^2 - 2r)\mathcal{L}^2] + \frac{4bCr^{-(b+1)}}{(1+Cr^{-b})^3}[(4r^3 + 2a^2r + 2a^2)\mathcal{E}^2 \\ + 4a\mathcal{E}\mathcal{L} - (2r - 2)\mathcal{L}^2] + \frac{1}{(1+Cr^{-b})^2}[(12r^2 + 2a^2)\mathcal{E}^2 \\ - 2\mathcal{L}^2] - (12r^2 - 12r + 2a^2) \quad (34)$$

respectively. After some long but straightforward deductions, we can get

$$\Delta r_{ms} = \frac{2b(b+3)r_0^{-b}(r_0^2 - 2r_0 + a^2) - 4br^{-b}(4r_0^2 - 6r_0 + 2a^2) - 6r_0^2\mathcal{E}_1^2}{4 + 12r_0\mathcal{E}_0^2 + 12r_0^2\mathcal{E}_0\mathcal{E}'_0 - 12r_0}, \quad (35)$$

where  $r_0$  and  $\mathcal{E}_0$  are expressed as

$$r_0 = 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}, \quad (36)$$

and

$$\mathcal{E}_0 = \frac{r_0^{3/2} - 2r_0^{1/2} + a}{r_0^{3/4}(r_0^{3/2} - 3r_0^{1/2} + 2a)^{1/2}}, \quad (37)$$

respectively, where

$$Z_1 \equiv 1 + (1 - a^2)^{1/3} [(1 + a)^{1/3} + (1 - a)^{1/3}],$$

$$Z_2 \equiv (3a^2 + Z_1^2)^{1/2}.$$

$\mathcal{E}'_0$  is the derivative of  $\mathcal{E}_0$  with  $r_0$ .  $\mathcal{E}_1^2$  is the correction term of  $\mathcal{E}_0^2$  with order  $O(C)$ .

The behavior of the correction term is similar to that of the Schwarzschild case. This can be seen in Fig. 2. As an example, Fig. 3 shows the dependence of the marginally stable circular orbit on  $b$  and  $C$  in Kerr spacetime with  $a = 0.5$ .

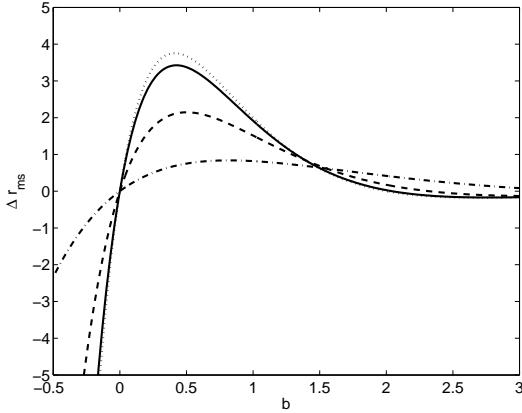


FIG. 2: The dependence of the correction term  $\Delta r_{ms}$  on the power-law index  $b$  and spin parameter  $a$ . Dotted line:  $a = 0$ ; solid line:  $a = 0.1$ ; dashed line:  $a = 0.5$ ; dash-dotted line:  $a = 0.9$ .

#### IV. DISCUSSION

The marginally stable circular orbit can be treated as the inner boundary of an accretion disk around a black hole in some cases. This is important because it is crucial to study the structure of accretion disks. It is also important when calculating the radiation of a standard

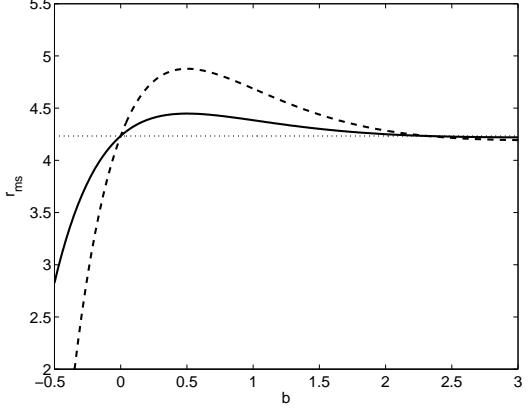


FIG. 3: The dependence of the marginally stable circular orbit  $r_{ms}$  in Kerr spacetime with  $a = 0.5$  on the power-law index  $b$  in the case of  $C < 1$ . The solid line and dashed line are for  $C = 0.1$  and  $C = 0.3$ , respectively. The dotted horizontal line represents  $r_{ms} = 4.233$  in the test particle case for  $a = 0.5$ .

accretion disk, where it is usually assumed that there is no radiation from the region inside the marginally stable circular orbit.

However, the widely accepted marginally stable circular orbit is based on the test particle assumption. When the temperature in an accretion disk is not very high (namely the thermal energy is much smaller than rest energy), the expression of the marginally stable circular orbit for a test particle is accurate enough. But when the temperature is so high that the thermal energy is non-negligible comparing to the rest energy, pressure may introduce some corrections to the location of the marginally stable circular orbit.

If the torque free boundary condition is applied at the inner boundary of accretion disks, the temperature near the marginally stable circular orbit would decrease with the decreasing of radius, which is different from the region far from the inner boundary. Assuming the dependence of temperature on radius near the marginally stable circular orbit is a power-law  $T \propto r^{-b}$ , then  $b < 0$ . As can be seen from Fig. 2, the correction term changes significantly with  $b$  when  $b$  is negative. As an example, for the Schwarzschild case, when  $C = 0.2$  and  $b = -0.2$ , the marginally stable circular orbit is  $r_{ms} = 5$ .

The correction to the marginally stable circular orbit not only influence the inner boundary of accretion disks, but also affect the energy release of the accretion flow around black holes. In Schwarzschild spacetime, the

binding energy  $E$  (see Eq. 9) at a certain radius  $r$  is [1]

$$E = \frac{r - 2}{r^{1/2}(r - 3)^{1/2}}, \quad (38)$$

which equals to 0.9428 at  $r = r_{ms} = 6$ , so the corresponding efficiency is  $1 - 0.9428 = 5.72\%$ . If the marginally stable circular orbit is different, the efficiency can be different, e.g., at  $r = 5$  the efficiency is 5.13%. In Kerr spacetime, the energy  $E$  of a test particle co-rotating in a circular orbit at  $r$  is described by Eq. (9). The dependence of the efficiency  $1 - E$  on radius  $r$  is shown in Fig. 4. Note that it is not monotonic, when the radius  $r < r_{ms}$ , the efficiency is also lower than that at the marginally stable circular orbit.

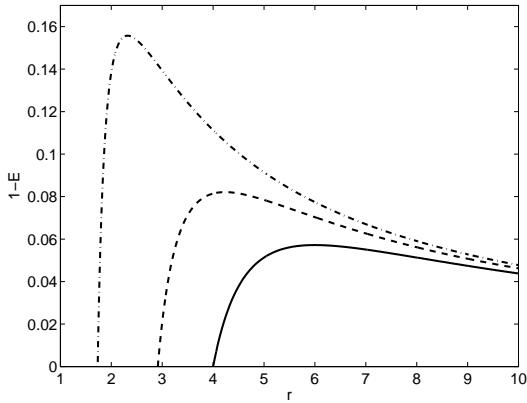


FIG. 4: The dependence of the efficiency  $1 - E$  on radius  $r$ . The solid line, dashed line and dash-dotted line correspond to the cases of  $a = 0, 0.5$  and  $0.9$ , respectively.

However, not only the true location of inner boundary of accretion disks is different from marginally stable circular orbit as shown by the analytic models [11–13], but as [17] also mentioned, in a disk where the advection is important, it is not proper to use the marginally stable circular orbit as its inner boundary to calculate the radiation of the disk. The reason is that in an advection dominated flow, there is no "equilibrium" of the circular motion, and the condition based on which the marginally stable circular orbit is derived is not fulfilled. But the widely used technique to measure black hole spin by fitting thermal soft state spectra [8, 18] is all right because in a standard thin disk, the marginally stable orbit is almost the same as for the test particle case, which is used for the measurement of the black hole spin parameters in X-ray binaries.

One thing should be noted is that our treatment is based on a perfect fluid disk. However, a real accretion disk consists of viscous fluid. Another point that may worth further consideration is the influence of pressure on the "abruptness" of the transition from inspiral to plunge regions. Because this property will heavily affect the inner boundary condition of accretion disks, if the transition is somehow less "abrupt", the fluid disk may continue further in beyond the marginally stable circular

orbit. However, this is out of the scope of this work. We hope to tackle this problem in a future work.

## Acknowledgments

This work is supported by the NSFC grant 10525113 and the 973 program 2007CB815405. Q.L. would like to thank all the members of AGN group in PKU for helpful discussions.

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